STATISTICAL MODEL OF TURBULENT FLOW WITH POLYMER ADDITIVES

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A closed equation is obtained for the function describing the length scale distribution of the turbulent velocity fluctuation energy in an isotropic flow of dilute linear high-molecular polymer solution.

Introduction

The following is an attempt to construct a theoretical model of the mechanism of action of polymer additives on the spectral structure of a turbulent flow. We have chosen for study the function $P_t(r)$, which describes the length scale distribution of the velocity fluctuation energy in an isotropic turbulent flow.

The presence of polymer additives modifies the shape of the two-point characteristics, correlation functions or spectra, as compared with that observed in an ordinary Newtonian fluid, as directly or indirectly indicated by the results of a series of experimental studies (see, e.g., [1-6]). The changes in these functions are attributable to the selective action of small amounts of added polymer on the velocity fluctuations with different length and time scales.

In deriving a closed equation for some two-point function in which the effect of polymer additives is to be taken into account it is necessary to have a clear qualitative picture of the mechanism of interaction of the turbulent velocity fluctuations and the polymer molecules dissolved in the fluid. Here, it is proposed to use Lumley's mechanism [7, 8]. The essence of this mechanism, which we will use for writing the initial dynamical equation for the velocity field, can be explained briefly as follows.

If the Reynolds number is high enough, there will exist in the turbulent flow a fluctuating field of the velocity gradient tensor $\hat{Z}$, which can be represented in the form of a sum of symmetric $\hat{S}$ and antisymmetric $\hat{\Omega}$ parts. Here, $\hat{S}$ is the strain rate tensor, and $\hat{\Omega}$ is the vorticity tensor. Each point of the turbulent flow is characterized by a certain relation between the quantities $S = |\hat{S}|$ and $\Omega = |\hat{\Omega}|$. When at some point in the flow the relation [7]

$$S^2 - \Omega^2 > \frac{1}{4T^2} \tag{1}$$

is realized, where T is the characteristic relaxation time of the polymer molecule, the polymer molecules present at that point are heavily stretched and increase in size and the viscosity of the fluid increases. (The theoretical and experimental research into the behavior of polymer molecules and the corresponding changes in viscosity under various hydrodynamic conditions is reviewed in [9].) Thus, the turbulent velocity field "senses" the presence of polymer additives at certain random points in the flow where the stretching criterion (1) is satisfied through the sharp increase in the viscosity at those points. Considering the physical significance of the tensors $\hat{S}$ and $\hat{\Omega}$ it can be said that in a turbulent flow with polymer additives the viscosity is selective with respect to the form of the motion: it varies only weakly in the presence of rotational motion and at those points at which simple shear ($S = \Omega$) is realized, and increases sharply when the motion involves tension. This selective change in viscosity leads to a marked change in the structure of the turbulent flow – to a restructuring of the length scale spectrum and the velocity field fluctuation probability distribution. As a result of this restructuring the flow must become less dissipative, and maintaining a given turbulent energy level requires less energy than in the case of a flow without polymer additives. This is how the proposed model of the interaction between polymer additives and an isotropic turbulent flow and the results of that interaction can be depicted.

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At the wall a similar mechanism leads to drag reduction [7, 8]. This idea is used as the basis for phenomenological models of turbulence that take the Toms effect into account [10].

We have attempted to derive analytically an expression for the term that takes the effect of polymer additives into account in the expression for the function $P_t(r)$ in the case of an isotropic flow with polymer molecules uniformly distributed in space. The derivation is based on the Navier–Stokes equation with a viscosity coefficient $\nu(x)$ that depends on the relation between S and Ω in accordance with the formula

$$\hat{\nu}(\mathbf{x}) = \nu + \nu_p \Theta \left[S^2(\mathbf{x}) - \Omega^2(\mathbf{x}) - \frac{1}{4T^2} \right]. \tag{2}$$

Here, ν is the viscosity of the solvent, ν_p is the longitudinal viscosity of the solution, i.e., the viscosity at those points at which the polymer molecules are stretched; $\Theta(\tau)$ is the Heaviside function; and

$$\hat{S}(\mathbf{x}) = \frac{1}{2}\left[\frac{\partial u_m(\mathbf{x})}{\partial x_n} + \frac{\partial u_n(\mathbf{x})}{\partial x_m}\right], \quad \hat{\Omega}(\mathbf{x}) = \frac{1}{2}\left[\frac{\partial u_m(\mathbf{x})}{\partial x_n} - \frac{\partial u_n(\mathbf{x})}{\partial x_m}\right]. \tag{3}$$

The viscosity coefficient model (2) ensures the inclusion of an anomalously high viscosity force only at those points of the flow at which the criterion (1) is satisfied. In this model the properties of the solution are expressed in terms of two parameters: ν_p and T, which are different for different polymer solutions. The concentration of the polymer molecules in the solution exerts an effect only through the viscosity coefficient ν_p.

Structure of the Equation for $P_t(r)$ in the Case of a Turbulent Flow with Polymer Additives

The function $P_t(r)$ is related with the longitudinal velocity field correlation function B(r, t) by the equation [11]

$$P_t(r) = -\frac{\partial}{\partial r} B(r, t). \tag{4}$$

The physical significance of the function $P_t(r)$ as the length scale distribution of turbulent fluctuation energy is apparent from the equation

$$\int_0^\infty P_t(r)\,dr = B(0, t) = \frac{2}{3} q(t), \tag{5}$$

where q(t) is the average energy of the turbulent velocity fluctuations.

We will first obtain the equation for the correlation function B(r, t) and then, using Eq. (4), go over to the equation for the function $P_t(r)$. As the initial dynamical equations we will use the Navier–Stokes equation for the velocity fluctuations, which, when the average velocity is equal to zero, has the form:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_k)}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left[\hat{\nu}(\mathbf{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right] + f_i, \quad i = 1, 2, 3, \tag{6}$$

and the incompressibility equation for the velocity fluctuations

$$\frac{\partial u_\beta}{\partial x_\beta} = 0. \tag{7}$$

In Eq. (6) p is the pressure fluctuations, ρ is the density of the solution, $\hat{\nu}(x)$ is the kinematic viscosity coefficient given by (2), and f_i is the vector random force field responsible for pumping the turbulent energy. The statistical properties of this field must be given. We will assume that the field is Gaussian and δ-correlated with respect to time [12].

The equation for the correlation tensor $B_{ij}(\mathbf{r}; t) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r})\rangle$ is derived in the usual way [13]. In going over to the equation for the correlation function B(r, t) we take into account the fact that in the case of isotropic turbulence in an incompressible fluid the two-point correlation tensors of the velocity and pressure fields are equal to zero, and the second and third order two-point velocity field tensors can be expressed in terms of scalar longitudinal correlation functions of the second B(r, t) and third $B_{LLL}(r, t)$ orders, respectively. Using corresponding formulas from [13] and taking into account the statistical properties of the random force field $f_i(\mathbf{x}, t)$ [12], we can obtain the following equation for B(r, t):

$$\frac{\partial B(r,\ t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) B_{LLL}(r, t) + 2\nu \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right) B(r,\ t) - \nu_p I(r,\ t) + \frac{2}{3}\varepsilon \exp\left(-\frac{r^2}{L^2}\right). \tag{8}$$

The last term in (8) describes the pumping of energy into the turbulent flow, ε is the pumping rate, and L is the length scale on which pumping takes place. The term I(r, t) in (8) is generated by the second term in expression (2) for the viscosity coefficient

$$I(r,\ t) = 2\left\{\frac{\partial}{\partial r_k} \left\langle \Theta \left[S^2(\mathbf{x}) - \Omega^2(\mathbf{x}) - \frac{1}{4T^2}\right] \left(\frac{\partial u_1(\mathbf{x})}{\partial x_k} + \frac{\partial u_k(\mathbf{x})}{\partial x_1}\right) u_1(\mathbf{x} + \mathbf{r}) \right\rangle\right\}\Bigg|_{\substack{r_1 = r \\ r_2 = r_3 = 0}}. \tag{9}$$

Acting on Eq. (8) by means of the operator $-\partial/\partial r$ and taking into account the definition (4), we obtain the equation for the function $P_t(r)$:

$$\begin{gathered} \frac{\partial P_t(r)}{\partial t} = -\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r} + \frac{4}{r}\right) B_{LLL}(r,\ t) + 2\nu\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}\right) P_t(r) + \\ + \nu_p \frac{\partial}{\partial r} I(r,\ t) + \frac{4}{3}\varepsilon \frac{r}{L^2} \exp\left(-\frac{r^2}{L^2}\right). \end{gathered} \tag{10}$$

In Eq. (10) the third-order velocity field correlation function $B_{LLL}(r, t)$, which describes the turbulent transfer of energy over the different length scales, must be expressed in terms of the known function $P_t(r)$. This can be done in the usual way, as in the case of an ordinary fluid without additives [14]. Then the equation for $P_t(r)$ takes the form:

$$\frac{\partial P_t(r)}{\partial t} = \frac{\partial}{\partial r}\left\{\left[2\nu + 2\gamma \int_0^r \sqrt{\tilde{r} P_t(\tilde{r})}\, d\tilde{r}\right] \left(\frac{\partial}{\partial r} + \frac{4}{r}\right) P_t(r)\right\} + \frac{4}{3}\varepsilon \frac{r}{L^2} \exp\left(-\frac{r^2}{L^2}\right) + \nu_p \frac{\partial}{\partial r} I(r,\ t). \tag{11}$$

Here, $\gamma = 0.24$ is a constant which can be related to the constant in Kolmogoroff's law of two-thirds if Eq. (11) is solved for the case $\nu_p = 0$ on the inertial range.

The function I(r, t) describes the action of the polymer additives on the structure of the function $P_t(r)$. The determination of the form of this function is also our principal task.

Relation between the Function I(r, t) and the Distribution Function

In order to carry out the averaging in expression (9) it is necessary to have the probability distribution function. Expression (9) contains nine velocity derivatives $\partial u_i/\partial x_j$, determined at the point x, and one velocity component u_1, determined at the point x+r. Averaging must be carried out only over those realizations of the velocity field which satisfy the criterion (1) both at the point x and at the point x+r. This means that out of the entire set of values of u_1 we will take into account only those that are realized with the satisfaction of condition (1) for the acceleration field at the point x+r. With this taken into account, the expression for I(r, t) in terms of the probability distribution function can be written in the form:

$$\begin{gathered} I(r,\ t) = 2\left\{\frac{\partial}{\partial r_k} \int\int\int\int\int\int\int\int\int\int \Theta\left(S^2 - \Omega^2 - \frac{1}{4T^2}\right)\left(\frac{\partial u_1}{\partial x_k} + \frac{\partial u_k}{\partial x_1}\right) u_1' \times \right. \\ \left. \times f^{(c)}\left(\left\{\frac{\partial u_i}{\partial x_j}\right\},\ u_1' \,\Big|\, S'^2 - \Omega'^2 > \frac{1}{4T^2}\right) d\left\{\frac{du_i}{dx_j}\right\} du_1'\right\}\Bigg|_{\substack{r_1 = r \\ r_2 = r_3 = 0}}. \end{gathered} \tag{12}$$

Here, $f^{(c)}(\ )$ is the two-point joint probability distribution function of the values of the derivatives $\partial u_i/\partial x_j$ and the velocity component u_1' taken subject to satisfaction of the inequality $S'^2 - \Omega'^2 > \frac{1}{4T^2}$. A prime denotes that the corresponding quantity relates to the point x+r.

The conditional function $f^{(c)}(\ )$ can be represented in the form of an integral of the unconditional joint probability distribution function $f\left(\left\{\frac{\partial u_i}{\partial x_j}\right\}, \left\{\frac{\partial u_i'}{\partial x_j'}\right\}, u_1'\right)$ of the values of the derivatives at the points x and x + r and the velocity component at the point x + r:

$$f^{(c)}\left(\left\{\frac{\partial u_i}{\partial x_j}\right\},\ u_1' \,\middle|\, S'^2 - \Omega'^2 > \frac{1}{4T^2}\right) = \tag{13}$$

$$= \iiiint\!\!\iiiint\!\!\int f\left(\left\{\frac{\partial u_i}{\partial x_j}\right\},\ \left\{\frac{\partial u_i'}{\partial x_j'}\right\},\ u_1'\right)\Theta\left(S'^2 - \Omega'^2 - \frac{1}{4T^2}\right) d\left\{\frac{\partial u_i'}{\partial x_j'}\right\}.$$

In accordance with Eqs. (3) we can obtain:

$$S^2 - \Omega^2 = \frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m}; \tag{14}$$

$$S'^2 - \Omega'^2 = \frac{\partial u_m'}{\partial x_n'}\frac{\partial u_n'}{\partial x_m'}. \tag{15}$$

Using the incompressibility condition (7), we can eliminate one velocity derivative from each of expressions (14) and (15) (e.g., $\partial u_3/\partial x_3$ and $\partial u'_3/\partial x'_3$). From this it follows that after the substitution of (13) in (12) it is possible to integrate with respect to the variables eliminated, leaving beneath the integral sign a 17-dimensional distribution function. In fact, there remain eight acceleration components at each point and one velocity component at the point x + r. The expression for I(r, t) can be written in the form:

$$I(r,\ t) = 2\left\{\frac{\partial}{\partial r_k}\int\overset{(17)}{\cdots}\int \Theta\left(F - \frac{1}{4T^2}\right)\Theta\left(\bar{F} - \frac{1}{4T^2}\right) a_k \xi_9 \times f(\{\xi_j\},\ \{\bar{\xi}_j\},\ \xi_9)\, d\{\xi_j\}\, d\{\bar{\xi}_j\}\, d\xi_9\right\}\Bigg|_{\substack{r_1=r\\ r_2=r_3=0}}. \tag{16}$$

Here, $\int\overset{(17)}{\cdots}\int d\{\xi_i\}d\{\bar{\xi}_i\}d\xi_9$ is a 17-dimensional integral, and $\{\xi_i\} = \{\xi_1, \xi_2, \ldots, \xi_8\}$, ξ_9, $\{\bar{\xi}_i\} = \{\xi_{10}, \xi_{11}, \ldots, \xi_{17}\}$ are the arguments of the 17-dimensional probability density function (PDF).

The correspondence between the variables $\{\xi_i\}$, $\{\bar{\xi}_i\}$, ξ_9 of the distribution function and the components of the derivatives $\{\partial u_i/\partial x_j\}$, $\{\partial u'_i/\partial x'_j\}$ and the velocity component u_1' can be given in the following form:

$\frac{\partial u_1}{\partial x_1}$	$\frac{\partial u_2}{\partial x_2}$	$\frac{\partial u_1}{\partial x_2}$	$\frac{\partial u_2}{\partial x_1}$	$\frac{\partial u_1}{\partial x_3}$	$\frac{\partial u_3}{\partial x_1}$	$\frac{\partial u_2}{\partial x_3}$	$\frac{\partial u_3}{\partial x_2}$	u_1'	$\frac{\partial u_2'}{\partial x_1'}$	$\frac{\partial u_2'}{\partial x_2'}$	$\frac{\partial u_1'}{\partial x_2'}$	$\frac{\partial u_2'}{\partial x_1'}$	$\frac{\partial u_1'}{\partial x_3'}$	$\frac{\partial u_3'}{\partial x_1'}$	$\frac{\partial u_2'}{\partial x_3'}$	$\frac{\partial u_3'}{\partial x_2'}$
ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}

The column matrix a_k can be defined in terms of the variables ξ_i:

$$a_k = \begin{vmatrix} 2\xi_1 \\ \xi_3 + \xi_4 \\ \xi_5 + \xi_6 \end{vmatrix}. \tag{17}$$

The expressions for F and $\bar{F}$ in terms of the variables $\{\xi_i\}$ and $\{\bar{\xi}_j\}$ have the form:

$$F = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 + \xi_3\xi_4 + \xi_5\xi_6 + \xi_7\xi_8, \tag{18}$$

$$\bar{F} = \xi_{10}^2 + \xi_{11}^2 + \xi_{10}\xi_{11} + \xi_{12}\xi_{13} + \xi_{14}\xi_{15} + \xi_{16}\xi_{17}.$$

The presence of two Heaviside Θ functions in the integrand expression in (16) imposes a limitation on the integration domain in the 16-dimensional space of the variables $\{\xi_i\}$, $\{\bar{\xi}_j\}$. This limitation represents the main difficulty in realizing the integration in (16) for any nontrivial choice of the shape of the 17-dimensional distribution function.

Shape of Two-Point 17-Dimensional PDF and Evaluation of Multidimensional Integral

In order to realize the integration in (16) it is necessary to make some concrete assumption concerning the shape of the 17-dimensional two-point PDF. We will assume that it is Gaussian. This choice is dictated, above all, by considerations of simplicity. It should be noted that taking the asymmetry of the distribution function into account does not make a contribution to I(r, t), since the integrand expression in (16) is even with respect to the integration variables ξ_i. Taking the fourth-order cumulants into account presents considerable computational difficulties. In this connection, it is still not clear whether in calculating I(r, t) the kurtosis of the PDF has any fundamental significance. Since our primary objective was to establish the decrease in turbulent energy dissipation resulting from the addition of small amounts of polymer, the choice of a Gaussian PDF should be quite adequate.

The Gaussian 17-dimensional two-point PDF can be defined as follows:

$$f(\{\xi_i\}) = \frac{\exp\left\{-\frac{1}{2}\sum_{\alpha=1}^{17}\sum_{\beta=1}^{17} d_{\alpha\beta}\,\xi_\alpha\,\xi_\beta\right\}}{(2\pi)^{17/2}\,D^{1/2}\sigma_1\sigma_2\ldots\sigma_{17}}\,. \tag{19}$$

Here, D is the determinant of the 17 × 17 correlation matrix d_{ik}, and d_{ik} is the inverse of the correlation matrix r_{ik}. The elements of r_{ik} are the correlations between the quantiites $\partial u_m/\partial x_n$ and $\partial u'_p/\partial x'_\ell$, the correlations of the quantities $\partial u_m/\partial x_n$ with each other and the quantities $\partial u'_m/\partial u'_n$ with each other, and the correlations of $\partial u_m/\partial x_n$ with u'_1 and $\partial u'_m/\partial x'_n$ with u_1'. Altogether there will be m = 136 correlations amenable to calculation in the matrix r_{ik}. It can be shown that all the matrix elements of r_{ik} can be expressed in terms of the unknown function $P_t(r)$ and its first and second derivatives with respect to the variable r. The elements of the inverse matrix can be calculated by Gauss's method [15]. All the values of the variance σ_i needed in (19) can be expressed in terms of the viscosity of the solvent ν, the dissipation rate $\bar{\varepsilon}$ and the turbulent energy in accordance with the formulas

$$\sigma_1 = \sigma_2 = \sigma_{10} = \sigma_{11} = \sqrt{\frac{\bar{\varepsilon}}{15\nu}},\quad \sigma_9 = \sqrt{\frac{2}{3}}\,q,$$
$$\sigma_i = \sqrt{\frac{2\bar{\varepsilon}}{15\nu}},\quad i = 3,\ 4,\ 5,\ 6,\ 7,\ 12,\ 13,\ 14,\ 15,\ 16,\ 17. \tag{20}$$

The presence of the viscosity coefficient of the solvent, rather than that of the solution, in (20) is attributable to the fact that on the viscous scale interval there is no stretching of the polymer molecules and the viscosity remains unchanged. This assertion will be proved by calculating the dependence of the function I(r, t) on the variable r.

The substitution of the explicit form of the PDF (19) in (16) makes it possible to realize all the operations of differentiation and evaluation of multidimensional integrals programmed on the right side. In the first place, it is possible to differentiate with respect to r_k and satisfy the condition $r_1 = r$, $r_2 = r_3 = 0$. Considering that the variable ξ_9 is not contained in the arguments of the Heaviside function, it is easy to evaluate the integral with respect to that variable. For integrating with respect to the variables $\{\xi_i\}$, $i = 1, 2, \ldots, 8$, and $\{\bar{\xi}_i\}$, $i = 10, 11, \ldots, 17$ it is convenient to use two sets of bipolar coordinates [16]. In this case the integrals can be calculated in explicit form with respect to all six angular variables in each set. In the expression obtained (a four-dimensional integral), by going over to two sets of polar coordinates it is possible to evaluate the integrals with respect to two more varibles. The remaining double integral reduces to an iterated integral. After performing all these rather cumbersome calculations, we arrive at the following expression for I(r, t):

$$I(r,\ t) = -G(r,\ t)\frac{1}{r}P_t(r), \tag{21}$$

where

$$G(r,\ t) = 10^{-2}\left[I_0(\omega_1)I_1(\omega_2) + I_1(\omega_1)I_0(\omega_2)\right]; \tag{22}$$

$$\omega_{\binom{1}{2}} = \frac{1}{3T^2\left[P_t'(0) \pm P_t'(r)\right]}\,. \tag{23}$$

Here, a prime denotes diferentiation with respect to the variable r:

$$I_k(\omega_i) = \frac{1}{\pi} \int\limits_{\omega_i}^{\infty} dx \exp\left(-\frac{x}{2}\right) \frac{\omega_i}{x^2} \left(1 - \frac{\omega_i}{x}\right)^{1/2} \left(\frac{5}{8} + \frac{3}{8}\frac{\omega_i}{x}\right)^{3/2+k} M_k(x),\ k = 0,\ 1;\ i = 1,\ 2. \tag{24}$$

In (24) we have used the notation:

$$M_0(x) = x^3 + 6x^2 + 24x + 48, \tag{25}$$

$$M_1(x) = x^4 + 8x^3 + 48x^2 + 192x + 384. \tag{26}$$

<u>Analysis of the Function I(r, t)</u>

With the expression for I(r, t) represented by Eqs. (21)-(26), Eq. (11) for the function $P_t(r)$ can be solved numerically, once the initial form of the function $P_0(r)$ and the values of the external parameters ν, ν_p, ε, L, and T are given. The initial function $P_0(r)$ can be given the form:

$$P_0(r) = \frac{2rB(0)}{L_0^2} \exp\left(-\frac{r^2}{L_0}\right)^2. \tag{27}$$

Here $B(0) = \sqrt{\frac{2}{3}q(0)}$, where q(0) is the initial value of the turbulent velocity field fluctuation energy, and L_0 is the characteristic length scale of the initial velocity field. Together with the viscosity coefficient of the solvent ν, the quantities q(0) and L_0 determine the initial value of the Reynolds number

$$\mathrm{Re}_0 = \frac{\sqrt{\frac{2}{3}q(0)}\,L_0}{\nu} \tag{28}$$

and the initial characteristic turbulent fluctuation time

$$T_0 = \frac{L_0}{\sqrt{\frac{2}{3}q(0)}}. \tag{29}$$

Together with Re_0, through the function G the quantity T_0 enters into Eq. (11) as an independent parameter. Therefore, a change in Re_0 may be accompanied by different regimes of variation of the characteristic time T_0, which will lead to different results.

We will assume that a change in the Reynolds number is achieved at the expense of a change in the characteristic fluctuation velocity $\sqrt{\frac{2}{3}q(0)}$ with L_0 fixed. Then, expressing $\sqrt{\frac{2}{3}q(0)}$ (28) in terms of the Reynolds number, for the dimensionless quantity $\hat{T}$ we obtain

$$\hat{T} = \frac{T}{T_0} = \frac{T}{\tau_0}\mathrm{Re}_0, \tag{30}$$

where

$$\tau_0 = \frac{L_0^2}{\nu} \tag{31}$$

represents the characteristic viscous dissipation time on the length macroscale. The time τ_0 is related to the characteristic turbulence time T_0 and the Kolmogoroff time scale by the following expressions:

$$\tau_0 = T_0 \mathrm{Re}_0 = T_K \mathrm{Re}_0^{3/2}. \tag{32}$$

In (23) we go over to the dimensionless variables:

$$\tau = \frac{T}{\tau_0};\ \rho = \frac{r}{L_0};\ \hat{P}'(\rho) = \frac{P'(r/L_0)}{T_0^2};\ \hat{\varepsilon} = \frac{\varepsilon L_0}{\left[\frac{2}{3}q(0)\right]^{3/2}} \tag{33}$$

and use the formula

$$\hat{P}_0{}'(0) = \frac{\mathrm{Re}_0}{15}(\hat{\varepsilon}_t + \hat{\varepsilon}), \tag{34}$$

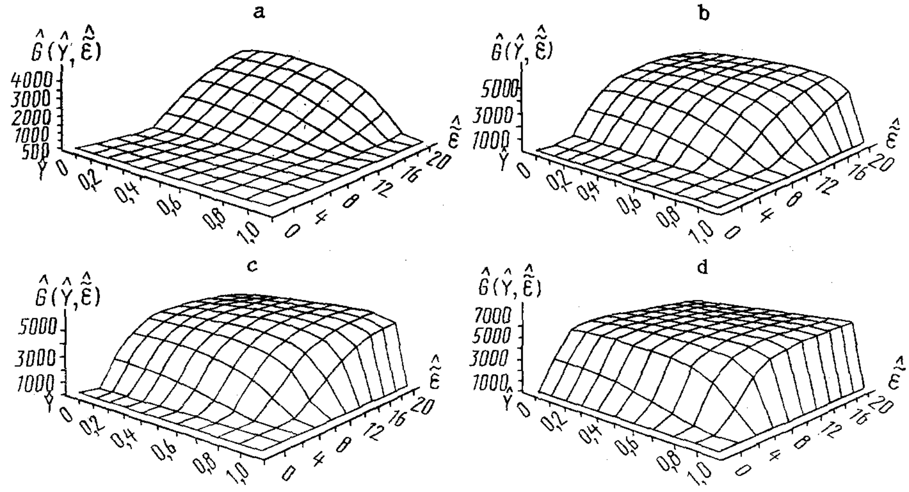

Fig. 1. Form of the function $G(Y, \hat{\tilde{\varepsilon}})$ calculated for $\tau = 0.001$ and various Reynolds numbers: (a) $\mathrm{Re}_0 = 10$; (b) 50; (c) 100; (d) 500.

where $\hat{\varepsilon}_t$ is the nondimensionalized rate of change of velocity fluctuation energy. This expression can be obtained from Eq. (11). Using (33) and (34), we can write the expression for $\omega_{\binom{1}{2}}$ in the form:

$$\omega_{\binom{1}{2}} = \frac{5}{\tau^2 \mathrm{Re}_0^3 \hat{\tilde{\varepsilon}} (1 + \hat{Y})}, \tag{35}$$

where

$$\hat{Y} = \frac{\hat{P}'(\rho)}{\hat{P}'(0)};\ \hat{\tilde{\varepsilon}} = \hat{\varepsilon}_t + \hat{\varepsilon}. \tag{36}$$

Since for the chosen polymer and solvent and the given scale L_0 the parameter τ is a concrete number, $\omega_{\binom{1}{2}}$ can be calculated for each value of the numbers Re_0, $\hat{\tilde{\varepsilon}}$, and $\hat{Y}$. From Eqs. (22)-(26) and (35) it is clear that the function $\hat{G}$ depends on the variable P only through $\hat{Y}$ and does not vary when $\hat{Y}$ changes sign:

$$\hat{G}(-\hat{Y}, \hat{\tilde{\varepsilon}}) = \hat{G}(\hat{Y}, \hat{\tilde{\varepsilon}}). \tag{37}$$

Therefore, for a given value of the parameter τ this function can be calculated for values of $\hat{Y}$ on the interval [0-1], for values of $\hat{\tilde{\varepsilon}}$ on the range (0-∞) and for various values of Re_0 and then used in solving Eqs. (11) by choosing those values of the function that correspond to the values of $\hat{Y}$ and $\hat{\tilde{\varepsilon}}$ realized at each specific point ρ and at the given moment of time t.

The form of the function $\hat{G}(\hat{Y}, \hat{\tilde{\varepsilon}})$ for various Re_0, calculated for $\tau = 10^{-3}$ ($T = 10^{-3}$ sec, $\nu = 10^{-6}$ m²/sec, $L_0 = 10^{-3}$ m), is shown in Fig. 1. It is clear from the figure that the values of $\hat{G}$ tend to zero as $\hat{Y}$ approaches unity, which corresponds to small values of ρ, i.e., to the viscous length scale interval. The function $\hat{G}$ also decreases with the dissipation rate $\hat{\tilde{\varepsilon}}$. As Re_0 increases, so does the rate of variation with respect to these variables. The specific values of the function $\hat{G}$ used in solving Eq. (11) depend on the chosen value of the polymer molecule concentration c_p. As the concentration increases from zero to large values, the values of the function $\hat{G}$ to be chosen are displaced towards smaller values of the dissipation rate.

It is clear from (35) that for very large values of Re_0 the quantities ω_1 and ω_2 will approach zero when $\hat{Y} \neq 1$ and $\hat{\tilde{\varepsilon}} \neq 0$. This involves an abrupt increase in the function $\hat{G}$ from zero to its maximum value, which, as may be seen from (22) and (24), is realized as $\omega_i \to 0$. As will be seen from the results of the calculations, when Re_0 is too large the effectiveness of the polymer additives is reduced.

For a given Re_0 the quantities ω_1 and ω_2 and, consequently, the function $\hat{G}$ depend on the parameter τ, which is equal to the ratio of the characteristic relaxation time of the polymer macromolecule T and the time scale of turbulence. For a small value of this ratio (very elastic polymer molecules or a large time scale τ_0) the values of ω_1 and ω_2 are large, which leads to a decrease in the function G and the diminution or total disappearance of the effect. When the value of the parameter T is too large ("soft" polymer molecules or very small scale of turbulence) the values of ω_1 and ω_2 will be small, and the function $\hat{G}$ will reach a maximum very sharply as $\hat{Y}$ deviates from unity, which will lead to an intensification of the effect.

The analysis of the expression for $\hat{G}(r, t)$ makes it possible to draw certain qualitative conclusions concerning the general nature of the last term in Eqs. (11). The function I(r, t), generated by the interaction of the polymer molecules and the turbulent velocity field, is equal to zero over the entire viscous length scale interval, where the function $P_t(r)$ is linear in the variable r. From the fact that the function I(r, t) vanishes as $r \to 0$, its negativeness at other values of r and, moreover, from the finiteness of the function I(r, t) for any values of r, which is evident from (22)-(24), it follows that the role of the last term in Eq. (11) reduces to the organization of turbulent energy transfer from the small to the large length scales. Counteracting the inertial term, it ensures a decrease in the total energy transfer from the large to the small length scales and thereby reduces the dissipativeness of the turbulent flow. From the fact that the function G(r, t) depends only on the derivative with respect to the variable r of the function $P_t(r)$, and not on the function itself, it follows that the interaction between the polymer molecules and the turbulence is determined by the acceleration field. In this case the structure of G(r, t) is such that energy transfer from the large to the small scales can only decrease, reverse mutual energy transfer being impossible.

From all that has been said concerning the structure of the last term of Eq. (11) it follows that its significance reduces to the renormalization of the expression describing the turbulent energy transfer over the length scale spectrum.

The properties of the statistical model proposed can be analyzed in detail on the basis of the solution of the closed Eq. (11). The results of solving this equation numerically in the steady-state case are presented in the next article.

NOTATION

$P_t(r)$, function describing the length scale distribution of the turbulent fluctuation energy; Z, velocity fluctuation gradient tensor; $\Theta(\tau)$, Heaviside unit function; B(r, t), longitudinal velocity field correlation function; $f_i \equiv f_i(x, t)$, random force vector field; I(r, t), function describing the action of the polymer additives on the structure of the function $P_t(r)$; T_0, characteristic time of the turbulent velocity fluctuations at t = 0; $\hat{\varepsilon}$, $\hat{\varepsilon}_t$, and $\hat{\tilde{\varepsilon}}$, nondimensionalized rates of injection, variation and dissipation of turbulent energy; $\hat{Y} = P'(\rho)/P'(0)$, $\tau = T/\tau_0$, $\tau_0 = L_0^2/\nu$, characteristic viscous dissipation time on the length scale L_0.

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MODEL OF STEADY ISOTROPIC FLOW WITH POLYMER ADDITIVES

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The results of numerically solving the equation for the function describing the length scale distribution of the turbulent velocity fluctuation energy in a steady isotropic flow of dilute linear high-molecular polymer solution are presented.

Introduction

In [1] a closed equation was derived for the function $P_t(r)$ describing the length scale distribution of the turbulent velocity fluctuation energy in an isotropic flow of dilute high-molecular polymer solution. In this equation (cf. (11) in [1]) what is new is the closed, i.e., fully expressed in terms of the unknown function $P_t(r)$ and the solution parameters, form of the term I(r, t), which takes into account the effect of the polymer additives on the structure of the turbulent flow. In deriving the expression for I(r, t) the mechanism of interaction of the polymer molecules and the turbulence described in [2, 3] was used. According to this mechanism, the viscosity of the dilute polymer solution is selective with respect to the nature of the fluid motion: it is not affected by rotational motion and increases sharply when the motion involves tension.

Formulation of the Steady-State Problem

With the expression for I(r, t), represented by Eqs. (21)-(26) in [1], Eq. (11) of [1] can be solved numerically, given the initial form of the function $P_0(r)$ and the parameters ν, ν_p, ε, L, and T. In this article we will consider only the steady-state case, when the function $P_t(r)$ does not depend on time, and the equilibrium in the flow is maintained by the balance between energy injection and dissipation. As will be seen from what follows, in the steady-state case the action of the polymer additives on the internal structure of the turbulent flow can be most graphically demonstrated.

In Eq. (11) [1] we pass to the limit as $t \to \infty$. Using the equalities

$$\lim_{t\to\infty} P_t(r) = P_\infty(r) \equiv P(r),$$
$$\lim_{t\to\infty} \frac{\partial P_t(r)}{\partial t} = 0, \quad \lim_{t\to\infty} I(r, t) \equiv I(r), \tag{1}$$

we can write the equation for P(r) in the form:

$$\frac{d}{dr}\left\{\left[2\nu + 2\gamma \int_0^r \sqrt{r'P(r')}\,dr'\right]\left(\frac{d}{dr} + \frac{4}{r}\right)P(r) + \nu_p I(r)\right\} = \frac{4}{3}\varepsilon \frac{r}{L^2}\exp\left(-\frac{r^2}{L^2}\right). \tag{2}$$

Integrating this equation with respect to r from 0 to r and using Eq. (21) from [1], together with the equations

$$I(0) = 0, \quad \varepsilon = 15\nu \left.\frac{dP(r)}{dr}\right|_{r=0}, \tag{3}$$

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